HALSEY L. ROYDEN | PATRICK M. FITZPATRICK

REAL ANALYSIS

5e

REAL ANALYSIS

Fifth Edition

Halsey L. Royden
Stanford University

Patrick M. Fitzpatrick
University of Maryland, College Park

Pearson

Content Management: *Jonathan Krebs, Jeff Weidenaar*
Content Production: *Manoj Lodhi*
Product Management: *Jessica Darczuk*
Rights and Permissions: *Anjali Singh*

Please contact https://support.pearson.com/getsupport/s/ with any queries on this content.

Library of Congress Cataloging-in-Publication Data

Names: Royden, H. L., author. | Fitzpatrick, Patrick, author.
Title: Real analysis / Halsey L. Royden, Stanford University, Patrick M. Fitzpatrick, University of Maryland, College Park.
Description: Fifth edition. | Hoboken, NJ : Pearson Education, Inc., [2023] | Includes bibliographical references and index.
Identifiers: LCCN 2022011423 | ISBN 9780137906529 | ISBN 0137906528
Subjects: LCSH: Functions of real variables. | Functional analysis. | Measure theory.
Classification: LCC QA331.5 .R6 2023 | DDC 515/.8—dc23/eng20220611
LC record available at https://lccn.loc.gov/2022011423

ScoutAutomatedPrintCode

Pearson

ISBN-10: 0-13-790652-8
ISBN-13: 978-0-13-790652-9

To the memory of my wife, Teresita Lega,

Patrick M. Fitzpatrick

To John Slavins

Halsey L. Royden

About the Authors

Halsey Royden was born in Pheonix, Arizona. He earned a BA from Stanford University at the age of 19, and one year later, an MA. After earning a PhD from Harvard University, he returned to Stanford to join the Department of Mathematics, where he remained for his professional career. He spent several sabbaticals at the Institute for Advanced Studies, Princeton. Between 1973 and 1981, he was dean of the School of Humanities and Sciences. During 1973–1974, he was a Guggenheim Fellow. The first edition of his *Real Analysis* was published in 1964. His research interests were in complex analysis and differential geometry.

Patrick Fitzpatrick was born in Youghal, Ireland. He studied at Rutgers University, where, at the age of 19 he earned an undergraduate degree and later a PhD. After post doctoral positions as a Courant Instructor at the Courant Institute, New York University, and an L. E. Dickson Instructor at the University of Chicago, he joined the Department of Mathematics at the University of Maryland, where he has remained for his professional career. He has spent several sabbaticals in Italy. During 1997–2007, he was chair of the Department of Mathematics. His book *Advanced Calculus* is published by the AMS. The first coauthored edition of *Real Analysis* was published in 2010. His research interests are in topological methods in nonlinear analysis.

Contents

Preface

The first three editions of Halsey L. Royden's *Real Analysis* contributed to the education of generations of mathematical analysis students. The fourth and this fifth edition of *Real Analysis* preserve the goal of its venerable predecessors—to present the measure theory, integration theory, and the elements of metric, topological, Hilbert, and Banach spaces that a modern analyst should know.

As in the preceding editions, in Part I, Lebesgue measure and integration for functions of a real variable are considered. In this fifth edition, the treatment of general measure and integration is placed in Part II rather than Part III. What was formerly in Part II is placed in Part III and a brief Part IV. In many courses based on this book, including my own, it has been found preferable to follow in the course the new ordering. This brings measure and integration on Euclidean space closer to their origin, the case of real variables. It also presents the opportunity to more strongly foreshadow, in the context of general measure and integration, concepts that later appear in general spaces.

First, a few remarks regarding specific topics in Part I.

- Somewhat simpler proofs of the Vitali Covering Lemma and Lebesgue's Theorem on the differentiability almost everywhere of a monotone function are provided.
- We prove von Neuman's Composition Theorem, according to which a strictly increasing, continuous function f: $[a, b] \to \mathbf{R}$ has an absolutely continuous inverse function if and only if the composition g: f is measurable whenever the function g: $\mathbf{R} \to \mathbf{R}$ is measurable.
- It is shown that a bounded function on a closed, bounded interval is Riemann integrable if and only if its set of discontinuities has measure zero. Alongside this, we present an ancestor of the Dominated Convergence Theorem for the Lebesgue integral, but for the Riemann integrable, called the Arzelá Convergence Theorem. The difficulty in proving this theorem without leaving the context of Riemann integration is remarkable.
- The concept of uniform integrability is prominently presented, and the Vitali Convergence Theorem is proven and made the centerpiece of the proof of the fundamental theorem of calculus for the Lebesgue integral. We prove that the divided difference functions for an absolutely continuous function are uniformly integrable, so that the fundamental theorem follows by directly taking the limit in its elementary, discrete formulation.
- Following Peter Lax, we consider rapidly Cauchy sequences in the $L^p(E)$ spaces: such sequences converge pointwise and in L^p to function in L^p. The identification of such sequences provides a more conceptual proof of the completeness of L^p spaces.
- An elegant proof of Lusin's Theorem, due to Peter Loeb and Eric Talvila, is given, and from this theorem it immediately follows that a measurable function is the pointwise limit almost everywhere of sequence of continuous functions. This is made the basis for proving, for $1 \leq p < \infty$, the separability of $L^p(\mathbf{R})$.
- The change of variables theorem for the Lebesgue integral for functions of a real variable is proven. This is one of many proofs in which the characterization of Lebesgue

measurable sets as being G_δ sets from which a set of measure zero has been excised is used. The proof brings to light delicate points regarding the measurability of compositions, which are informed by the just mentioned von Neumann Composition Theorem.

- A brief section on convergence in measure and convergence in the mean is included.

The treatment of Lebesgue measure and integration on $\mathbf{R}^n$ now includes the following.

- Convolution of pairs of functions on $\mathbf{R}^n$ are considered. First, Young's Convolution Inequality and Minkowski's Integral Inequality are proven. Based on these, we prove that, for $1 \le p < \infty$, the compactly supported, infinitely differentiable functions are dense in $L^p(\mathbf{R}^n, \mu_n)$, and also prove a smooth version of Urysohn's Lemma in $\mathbf{R}^n$.
- Sufficient conditions for a mapping on $\mathbf{R}^n$ to preserve Lebesgue measurability of sets are provided. Being Lipschitz is one such condition. We prove the Vitali Partition Theorem, according to which an open subset of R^n is, after the excision of a set of measure zero, the union of a disjoint, countable collection of open balls. Using this, it follows that a linear operator $L: R^n \to R^n$ preserves distance between points if and only is it preserves Lebesgue measure. This provides a simple geometric foundation for the proof that multiplication by the absolute value of the determinant gives the change in Lebesgue measure induced by a linear operator.
- We prove that any finite Borel measure on $\mathbf{R}^n$ is regular, in anticipation of the later proof of Ulam's Regularity Theorem, according to which a finite Borel measure on a separable, complete metric space is regular.
- Care has been taken to explicitly present the set-theoretic properties of measurable rectangles that are at the heart of the proof of the Fubini and Tonelli Theorems, and which, once presented, actually suggest the method of proof.

There is more likely to be agreement about what an analyst should know about measure and integration than there is about what should be known about general spaces. Historically, important special cases of theorems in general spaces were first revealed for spaces of integrable functions. We have commented on these generalizations as the special cases occur, with a view toward motivating them. Consider three examples of these. (i) An important consequence of the Hahn-Banach Theorem regarding the existence of bounded linear functionals on a normed linear space that separate points is explicitly established, and used, when considering, for $1 \le p < \infty$, the dual of the L^p spaces. (ii) In these same spaces, weak sequential compactness of closed, bounded, convex subsets is proven, and used to establish minima for convex, continuous functions: this foreshadows weak sequential compactness in reflexive Banach space. (iii) The Uniform Boundedness Principle is directly proved for linear functionals rather than as a consequence of the Baire Category Theorem, using an elegant proof of Hahn. This is used to show that weakly convergent sequences are bounded, in these same L^p spaces.

Regarding the selection of general spaces in Part III, normed linear spaces need little motivation, since the L^p spaces have already been broadly considered in the first two parts. Then there are important concepts regarding subsets of normed linear spaces that are independent of the ambient linear structure. These properties are captured in the structure of a metric space, and for these the concepts of compactness and completeness play leading roles. The importance of completeness is brought to the forefront in the Baire Category

Theorem, with its quite elementary proof, and its remarkable diverse applications to individual operators (the Open Mapping and Closed Graph Theorem), to sequences of operators and functions, both linear and non-linear, and to properties of set-functions that are limits of sequences of measures. For complete metric spaces, we also prove, again with quite elementary proofs, the Banach Contraction Principle and a corollary, the Picard Existence Theorem. The importance of compactness goes back to the proof at the beginning of calculus of Rolle's Theorem. We show that a metric space is compact if and only if every continuous real-valued function on it has a minimum value.

Perhaps, for the young analyst, the motivation to extend the concept of metric space to topological space is not so evident. However, we prove a theorem of Riesz which asserts that the closed unit ball of a normed linear space is compact with respect to the metric induced by the norm if and only if the space has finite dimension. We also show, in Part I, that sequential weak compactness is sometimes an able substitute for the loss of compactness for the metric induced by the norm. It is natural to seek the appropriate metric with respect to which convergence is weak sequential compactness. However, we prove that for an infinite dimensional normed linear space, there is not a metric with respect to which sequential convergence is weak sequential convergence. Topological spaces provide a more flexible structure that is not dependent on sequential arguments or countable constructions. For topological spaces, we prove two fundamental theorems, which are strong extensions of their forebears in metric spaces. For metric spaces, the countable product of such spaces is directly defined, and it follows immediately that the countable product of compact metric spaces is compact. The Tychonoff Product Theorem asserts that the arbitrary product of compact topological spaces is compact. The proof of Urysohn's Lemma in a metric space is an immediate consequence of the continuity of the distance functions. The proof of Urysohn's Lemma for a normal topological space is more delicate. It has many interesting applications, among them being the Urysohn Metrization Theorem: if a topological space has a countable base, then the topology is induced by a metric if and only if it is normal.

Selected topics in linear operator theory and, in particular, for linear operators on Hilbert spaces, are presented. These include the consideration of Fredholm operators, which are widely useful in both applied mathematics and modern topology. More particular specific topics, for instance, von Neumann's Theorem on the existence of Haar measure on a compact group and the Stone-Weierstrass Theorem, are also presented. These are both important and elegant. Nevertheless, a different selections of topics could also be well justified. I welcome comments regarding the selections which have, or should have, been made. And, of course, any other comments are also very welcome. I can be contacted through pmf@math.umd.edu. A list of errata and remarks is on https://www.pearsonhighered.com/mathstatsresources.

ACKNOWLEDGMENTS

Regarding the fourth edition, it was a pleasure to acknowledge my indebtedness to teachers, colleagues, and students. My undergraduate teachers, Professors John Bender and Silvio Aurora, revealed to me the beauty of mathematics and to the possibility that I could become a mathematician. A penultimate draft of the entire manuscript was read by Diogo Arsénio, whom I warmly thank for his observations and suggestions, which materially improved that draft. Here in my mathematical home, the University of Maryland, I have written notes for various analysis courses, which have been incorporated into the present edition. A number of students in my graduate analysis classes worked through parts of drafts of this edition, and their comments and suggestions have been most helpful: I thank Avner Halevy, J. J. Lee, Kevin McGoff, and Himanshu Tiagi. I am pleased to acknowledge a particular debt to Brendan Berg who created the index, proofread the final manuscript, and kindly refined my Tex skills. I have benefited from conversations with many friends and colleagues; in particular, with Diogo Arsénio, Stu Antman, Craig Evans, Manos Grillakis, Tim Healey, Brian Hunt, Jacobo Pejsachowicz, and Michael Renardy.

The fourth edition was finely read by Jose Renato Ramos Barbosa and by Richard Hevener, each of whom provided me with errata sheets and many excellent suggestions. Sam Punshon-Smith was an invaluable assistant in preparing the reprintings of that edition. Regarding the preparation of this fifth edition, I am endebted to Stephane Gilles for his generous, expert help during a rather difficult period.

I warmly thank the Publisher, and the reviewers: Randall Swift, California State Polytechnic University; Jianfeng Lu, Duke University; Jeffrey Neugebauer, Eastern Kentucky University; Karl Liechty, DePaul University; Jason Crann, Carleton University; John Harding, New Mexico State University.

Patrick M. Fitzpatrick
La Jolla, CA
May, 2022

This page intentionally left blank

PART ONE

LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

This page intentionally left blank

Preliminaries on Sets, Mappings, and Relations

Contents

In these preliminaries, we describe some notions regarding sets, mappings, and relations that will be used throughout the book. Our purpose is descriptive and the arguments given are directed toward plausibility and understanding rather than rigorous proof based on an axiomatic basis for set theory. There is a system of axioms called the Zermelo-Fraenkel Axioms for Sets upon which it is possible to formally establish properties of sets and thereby properties of relations and functions.

UNIONS AND INTERSECTIONS OF SETS

For a set A,[1] the membership of the element x in A is denoted by $x \in A$ and the nonmembership of x in A is denoted by $x \notin A$. We often say a member of A belongs to A and call a member of A a *point* in A. Frequently, sets are denoted by braces, so that $\{x \mid \text{statement about } x\}$ is the set of all elements x for which the statement about x is true.

Two sets are the same provided they have the same members. Let A and B be sets. We call A a **subset** of B provided each member of A is a member of B; we denote this by $A \subseteq B$ and also say that A is contained in B or B contains A. A subset A of B is called a **proper subset** of B provided $A \neq B$. The **union** of A and B, denoted by $A \cup B$, is the set of all points that belong either to A or to B; that is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. The word *or* is used here in the nonexclusive sense, so that points which belong to both A and B belong to $A \cup B$. The **intersection** of A and B, denoted by $A \cap B$, is the set of all points that belong to both A and B; that is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The **complement** of A in B, denoted by $B \sim A$, is the set of all points in B that are not in A; that is, $B \sim A = \{x \mid x \in B, x \notin A\}$. If, in a particular discussion, all of the sets are subsets of a reference set X, we often refer to $X \sim A$ simply as the complement of A.

The set that has no members is called the **empty-set** and denoted by $\emptyset$. A set that is not equal to the empty-set is called non-empty. We refer to a set that has a single member as a **singleton set**. Given a set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$ or 2^X; it is called the **power set** of X.

In order to avoid the confusion that might arise when considering sets of sets, we often use the words "collection" and "family" as synonyms for the word "set." Let $\mathcal{F}$ be a collection of sets. We define the union of $\mathcal{F}$, denoted by $\bigcup_{F \in \mathcal{F}} F$, to be the set of points

[1] The *Oxford English Dictionary* devotes several hundred pages to the definition of the word "set."

that belong to at least one of the sets in $\mathcal{F}$. We define the intersection of $\mathcal{F}$, denoted by $\bigcap_{F \in \mathcal{F}} F$, to be the set of points that belong to every set in $\mathcal{F}$. The collection of sets $\mathcal{F}$ is said to be **disjoint** provided the intersection of any two distinct sets in $\mathcal{F}$ is empty. For a family $\mathcal{F}$ of sets, the following identities are established by checking set inclusions.

De Morgan's identities

$$X \sim \left[\bigcup_{F \in \mathcal{F}} F\right] = \bigcap_{F \in \mathcal{F}} [X \sim F] \quad \text{and} \quad X \sim \left[\bigcap_{F \in \mathcal{F}} F\right] = \bigcup_{F \in \mathcal{F}} [X \sim F],$$

that is, the complement of the union is the intersection of the complements, and the complement of the intersection is the union of the complements.

For a set Λ, suppose that for each $\lambda \in \Lambda$, there is defined a set E_λ. Let $\mathcal{F}$ be the collection of sets $\{E_\lambda \mid \lambda \in \Lambda\}$. We write $\mathcal{F} = \{E_\lambda\}_{\lambda \in \Lambda}$ and refer to this as an **indexing** (or **parametrization**) of $\mathcal{F}$ by the **index set** (or **parameter set**) Λ.

Mappings between sets

Given two sets A and B, by a **mapping** or **function** from A into B we mean a correspondence that assigns to each member of A a member of B. In the case B is the set of real numbers we always use the word "function." Frequently we denote such a mapping by $f\colon A \to B$, and for each member x of A, we denote by $f(x)$ the member of B to which x is assigned. For a subset A' of A, we define $f(A') = \{b \mid b = f(a) \text{ for some member } a \text{ of } A'\}$: $f(A')$ is called the image of A' under f. We call the set A the **domain** of the function f and $f(A)$ the **image** or **range** of f. If $f(A) = B$, the function f is said to be **onto**. If for each member b of $f(A)$ there is exactly one member a of A for which $b = f(a)$, the function f is said to be **one-to-one**. A mapping $f\colon A \to B$ that is both one-to-one and onto is said to be **invertible**; we say that this mapping establishes a **one-to-one correspondence** between the sets A and B. Given an invertible mapping $f\colon A \to B$, for each point b in B, there is exactly one member a of A for which $f(a) = b$ and it is denoted by $f^{-1}(b)$. This assignment defines the mapping $f^{-1}\colon B \to A$, which is called the **inverse** of f. Two sets A and B are said to be **equipotent** provided there is an invertible mapping from A onto B. Two sets which are equipotent are, from the set-theoretic point of view, indistinguishable.

Given two mappings $f\colon A \to B$ and $g\colon C \to D$ for which $f(A) \subseteq C$ then the composition $g \circ f\colon A \to D$ is defined by $[g \circ f](x) = g(f(x))$ for each $x \in A$. It is not difficult to see that the composition of invertible mappings is invertible. For a set D, define the identity mapping $id_D\colon D \to D$ by $id_D(x) = x$ for all $x \in D$. A mapping $f\colon A \to B$ is invertible if and only if there is a mapping $g\colon B \to A$ for which

$$g \circ f = id_A \text{ and } f \circ g = id_B.$$

Even if the mapping $f\colon A \to B$ is not invertible, for a set E, we define $f^{-1}(E)$ to be the set $\{a \in A \mid f(a) \in E\}$; it is called the **inverse image** of E under f. We have the following useful properties: for any two sets E_1 and E_2,

$$f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2),\ f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$$

and

$$f^{-1}(E_1 \sim E_2) = f^{-1}(E_1) \sim f^{-1}(E_2).$$

Finally, for a mapping $f\colon A \to B$ and a subset A' of its domain A, the **restriction** of f to A', denoted by $f|_{A'}$, is the mapping from A' to B which assigns $f(x)$ to each $x \in A'$.

EQUIVALENCE RELATIONS, THE AXIOM OF CHOICE, AND ZORN'S LEMMA

Given two non-empty sets A and B, the **Cartesian product** of A with B, denoted by $A \times B$, is defined to be the collection of all ordered pairs (a, b), where $a \in A$ and $b \in B$ and we consider $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.[2] For a non-empty set X, we call a subset R of $X \times X$ a **relation** on X and write $x\,R\,x'$ provided (x, x') belongs to R. The relation R is said to be **reflexive** provided $x\,R\,x$, for all $x \in X$; the relation R is said to be **symmetric** provided $x\,R\,x'$ if $x'\,R\,x$; the relation R is said to be **transitive** provided whenever $x\,R\,x'$ and $x'\,R\,x''$, then $x\,R\,x''$.

Definition *A relation R on a set X is called an* **equivalence relation** *provided it is reflexive, symmetric, and transitive.*

Given an equivalence relation R on a set X, for each $x \in X$, the set $R_x = \{x' \mid x' \in X,\ x\,R\,x'\}$ is called the **equivalence class** of x (with respect to R). The collection of equivalence classes is denoted by X/R. For example, given a set X, the relation of equipotence is an equivalence relation on the collection 2^X of all subsets of X. The equivalence class of a set with respect to the relation equipotence is called the **cardinality** of the set.

Let R be an equivalence relation on a set X. Since R is symmetric and transitive, $R_x = R_{x'}$ if and only if $x\,R\,x'$ and therefore the collection of equivalence classes is disjoint. Since the relation R is reflexive, X is the union of the equivalence classes. Therefore, X/R is a disjoint collection of non-empty subsets of X whose union is X. Conversely, given a disjoint collection $\mathcal{F}$ of non-empty subsets of X whose union is X, the relation of belonging to the same set in $\mathcal{F}$ is an equivalence relation R on X for which $\mathcal{F} = X/R$.

Given an equivalence relation on a set X, it is often necessary to choose a subset C of X which consists of exactly one member from each equivalence class. Is it obvious that there is such a set? Ernst Zermelo called attention to this question regarding the choice of elements from collections of sets. Suppose, for instance, we define two real numbers to be rationally equivalent provided their difference is a rational number. It is easy to check that this is an equivalence relation on the set of real numbers. But it is not easy to identify a set of real numbers that consists of exactly one member from each rational equivalence class.

Definition *Let $\mathcal{F}$ be a non-empty family of non-empty sets. A* **choice function** *f on $\mathcal{F}$ is a function f from $\mathcal{F}$ to $\cup_{F \in \mathcal{F}} F$ with the property that for each set F in $\mathcal{F}$, $f(F)$ is a member of F.*

Zermelo's Axiom of Choice *Let $\mathcal{F}$ be a non-empty collection of non-empty sets. Then there is a choice function on $\mathcal{F}$.*

Very roughly speaking, a choice function on a family of non-empty sets "chooses" a member from each set in the family. We have adopted an informal, descriptive approach to set theory and accordingly we will freely employ, without further ado, the Axiom of Choice.

[2] In a formal treatment of set theory based on the Zermelo-Fraenkel Axioms, an ordered pair (a, b) is defined to be the set $\{\{a\}, \{a, b\}\}$ and a function with domain in A and image in B is defined to be a non-empty collection of ordered pairs in $A \times B$ with the property that if the ordered pairs (a, b) and (a, b') belong to the function, then $b = b'$.

Definition *A relation R on a set non-empty X is called a* **partial ordering** *provided it is reflexive, transitive, and, for x, x' in X,*

$$\text{if } x\,R\,x' \text{ and } x'\,R\,x, \text{ then } x = x'.$$

A subset E of X is said to be **totally ordered** *provided for x, x' in E, either $x\,R\,x'$ or $x'\,R\,x$. A member x of X is said to be an* **upper bound** *for a subset E of X provided $x'Rx$ for all $x' \in E$, and said to be* **maximal** *provided the only member x' of X for which $x\,R\,x'$ is $x' = x$.*

For a family $\mathcal{F}$ of sets and $A, B \in \mathcal{F}$, define $A\,R\,B$ provided $A \subseteq B$. This relation of **set inclusion** is a partial ordering of $\mathcal{F}$. Observe that a set F in $\mathcal{F}$ is an upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every set in $\mathcal{F}'$ is a subset of F and a set F in $\mathcal{F}$ is maximal provided it is not a proper subset of any set in $\mathcal{F}$. Similarly, given a family $\mathcal{F}$ of sets and $A, B \in \mathcal{F}$ define $A\,R\,B$ provided $B \subseteq A$. This relation of **set containment** is a partial ordering of $\mathcal{F}$. Observe that a set F in $\mathcal{F}$ is an upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every set in $\mathcal{F}'$ contains F and a set F in $\mathcal{F}$ is maximal provided it does not properly contain any set in $\mathcal{F}$.

Zorn's Lemma *Let X be a partially ordered set for which every totally ordered subset has an upper bound. Then X has a maximal member.*

We will use Zorn's Lemma to prove some of our most important results, including the Hahn-Banach Theorem, the Tychonoff Product Theorem, and the Krein-Milman Theorem. Zorn's Lemma is equivalent to Zermelo's Axiom of Choice. In the book Functional Analysis by Theo Bühler and Deitmar Salamon, there is a discussion and concise proof of the equivalence of the Axiom of Choice and Zorn's Lemma.

We have defined the Cartesian product of two sets. It is useful to define the Cartesian product of a general parametrized collection of sets. For a collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ parametrized by the set Λ, the Cartesian product of $\{E_\lambda\}_{\lambda \in \Lambda}$, which we denote by $\Pi_{\lambda \in \Lambda} E_\lambda$, is defined to be the set of functions f from Λ to $\bigcup_{\lambda \in \Lambda} E_\lambda$ such that for each $\lambda \in \Lambda$, $f(\lambda)$ belongs to E_λ. It is clear that the Axiom of Choice is equivalent to the assertion that the Cartesian product of a non-empty family of non-empty sets is non-empty. Note that the Cartesian product is defined for a parametrized family of sets and that two different parametrizations of the same family will have different Cartesian products. This general definition of Cartesian product is consistent with the definition given for two sets. Indeed, consider two non-empty sets A and B. Define $\Lambda = \{\lambda_1, \lambda_2\}$ where $\lambda_1 \neq \lambda_2$ and then define $E_{\lambda_1} = A$ and $E_{\lambda_2} = B$. The mapping that assigns to the function $f \in \Pi_{\lambda \in \Lambda} E_\lambda$ the ordered pair $(f(\lambda_1), f(\lambda_2))$ is an invertible mapping of the Cartesian product $\Pi_{\lambda \in \Lambda} E_\lambda$ onto the collection of ordered pairs $A \times B$ and therefore these two sets are equipotent. For two sets E and Λ, define $E_\lambda = E$ for all $\lambda \in \Lambda$. Then the Cartesian product $\Pi_{\lambda \in \Lambda} E_\lambda$ is equal to the set of all mappings from Λ to E and is denoted by E^Λ.

C H A P T E R 1

The Real Numbers: Sets, Sequences, and Functions

Contents

We assume the reader has a familiarity with the properties of real numbers, sets of real numbers, sequences of real numbers, and real-valued functions of a real variable, which are usually treated in an undergraduate course in analysis. This familiarity will enable the reader to assimilate the present chapter, which is devoted to rapidly but thoroughly establishing those results which will be needed and referred to later. We assume that the set of real numbers, which is denoted by $\mathbf{R}$, satisfies three types of axioms. We state these axioms and derive from them properties on the natural numbers, rational numbers, and countable sets. With this as background, we establish properties of open and closed sets of real numbers; convergent, monotone, and Cauchy sequences of real numbers; and continuous real-valued functions of a real variable.

1.1 THE FIELD, POSITIVITY, AND COMPLETENESS AXIOMS

We assume as given the set $\mathbf{R}$ of real numbers such that for each pair of real numbers a and b, there are defined real numbers $a+b$ and ab called the sum and product, respectively, of a and b for which the following Field Axioms, Positivity Axioms, and Completeness Axiom are satisfied.

The field axioms

Commutativity of Addition: For all real numbers a and b,

$$a+b=b+a.$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a+b)+c=a+(b+c).$$

The Additive Identity: There is a real number, denoted by 0, such that

$$0+a=a+0=a \qquad \text{for all real numbers } a.$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0.$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba.$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc).$$

The Multiplicative Identity: There is a real number, denoted by 1, such that

$$1a = a1 = a \qquad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

The Nontriviality Assumption:

$$1 \neq 0.$$

Any set that satisfies these axioms is called a **field**. It follows from the commutativity of addition that the additive identity, 0, is unique, and we infer from the commutativity of multiplication that the multiplicative unit, 1, also is unique. The additive inverse and multiplicative inverse also are unique. We denote the additive inverse of a by $-a$ and, if $a \neq 0$, its multiplicative inverse by a^{-1} or $1/a$. If we have a field, we can perform all the operations of elementary algebra, including the solution of simultaneous linear equations. We use the various consequences of these axioms without explicit mention[1].

The positivity axioms

In the real numbers there is a natural notion of order: greater than, less than, and so on. A convenient way to codify these properties is by specifying axioms satisfied by the set of positive numbers. There is a set of real numbers, denoted by $\mathcal{P}$, called the set of **positive numbers.** It has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are also positive.
P2 For a real number a, exactly one of the following three alternatives is true:

$$a \text{ is positive}, \qquad -a \text{ is positive}, \qquad a = 0.$$

[1] A systematic development of the consequences of the Field Axioms may be found in the first chapter of the classic book *A Survey of Modern Algebra* by Garrett Birkhoff and Saunders MacLane [BM97].

The Positivity Axioms lead in a natural way to an ordering of the real numbers: for real numbers a and b, we define $a > b$ to mean that $a - b$ is positive, and $a \geq b$ to mean that $a > b$ or $a = b$. We then define $a < b$ to mean that $b > a$, and $a \leq b$ to mean that $b \geq a$.

Using the Field Axioms and the Positivity Axioms, it is possible to formally establish the familiar properties of inequalities (see Problem 2). Given real numbers a and b for which $a < b$, we define $(a, b) = \{x \mid a < x < b\}$, and say a point in (a, b) lies between a and b. We call a non-empty set I of real numbers an **interval** provided for any two points in I, and all the points that lie between these points also belong to I. Of course, the set (a, b) is an interval, as are the following sets:

$$[a, b] = \{x \mid \ a \leq x \leq b\}\,; [a, b) = \{x \mid \ a \leq x < b\}\,; (a, b] = \{x \mid \ a < x \leq b\}\,. \qquad (1)$$

The completeness axiom

A non-empty set E of real numbers is said to be **bounded above** provided there is a real number b such that $x \leq b$ for all $x \in E$: the number b is called an **upper bound** for E. Similarly, we define what it means for a set to be **bounded below** and for a number to be a **lower bound** for a set. A set that is bounded above need not have a largest member. But the next axiom asserts that it does have a smallest upper bound.

The Completeness Axiom *Let E be a non-empty set of real numbers that is bounded above. Then among the set of upper bounds for E there is a smallest, or least, upper bound.*

For a non-empty set E of real numbers that is bounded above, the **least upper bound** of E, the existence of which is asserted by the Completeness Axiom, will be denoted by l.u.b. E. The least upper bound of E is usually called the **supremum** of E and denoted by $\sup E$. It follows from the Completeness Axiom that every non-empty set E of real numbers that is bounded below has a **greatest lower bound**; it is denoted by g.l.b. E and usually called the **infimum** of E and denoted by $\inf E$. A non-empty set of real numbers is said to be **bounded** provided it is both bounded below and bounded above.

The triangle inequality

We define the **absolute value** of a real number x, $|x|$, to be x if $x \geq 0$ and to be $-x$ if $x < 0$. The following inequality, called the **Triangle Inequality**, is fundamental in mathematical analysis: for any pair of real numbers a and b,

$$|a + b| \leq |a| + |b|.$$

The extended real numbers

It is convenient to introduce the symbols ∞ and $-\infty$ and write $-\infty < x < \infty$ for all real numbers x. We call the set $\mathbf{R} \cup \pm\infty$ the **extended real numbers**. If a non-empty set E of real numbers is not bounded above we define its supremum to be ∞. It is also convenient to define $-\infty$ to be the supremum of the empty-set. Therefore, every set of real numbers has a supremum that belongs to the extended real numbers. Similarly, we can extend the concept