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CALCULUS

CONCEPTS & CONTEXTS

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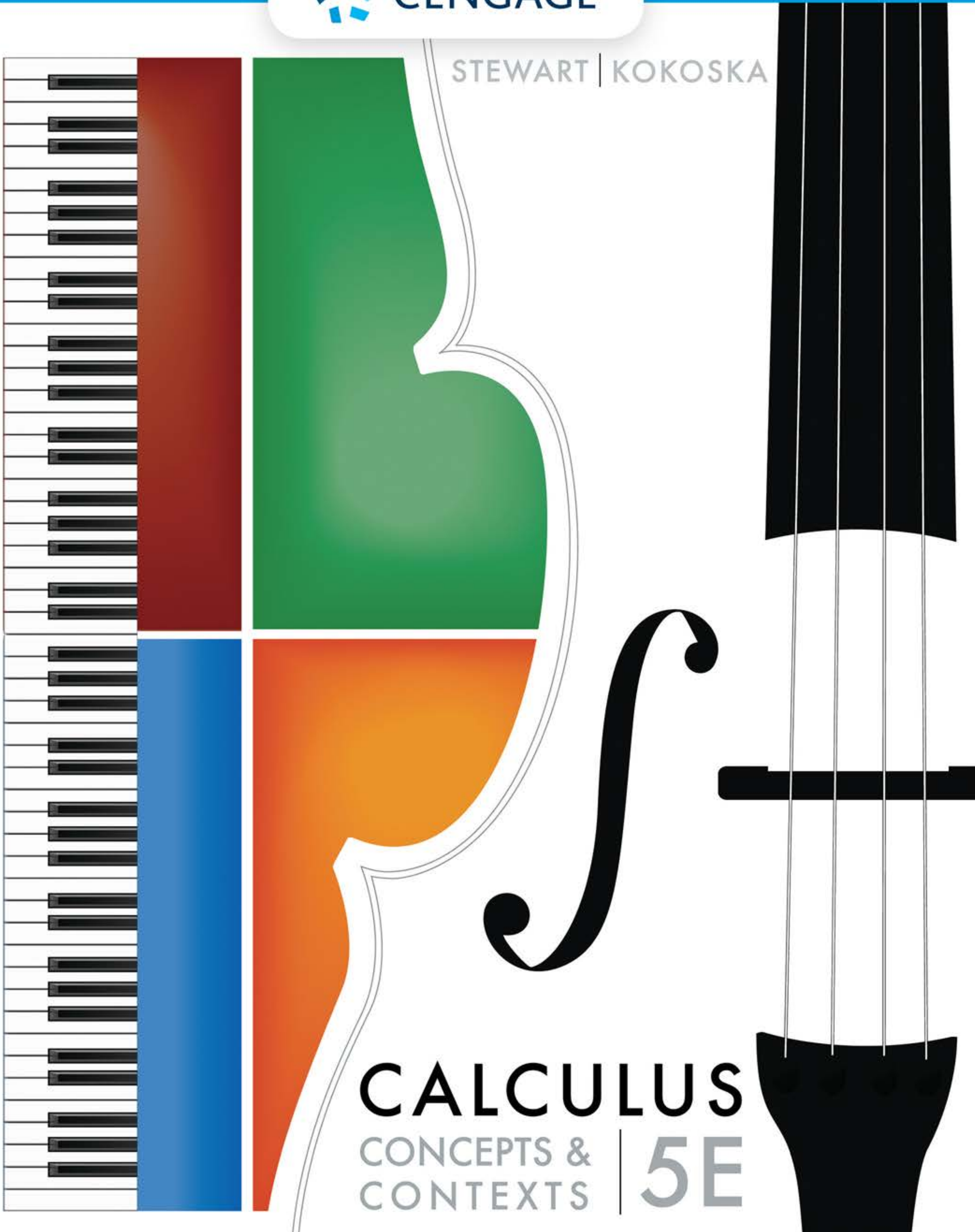

Calculus

Concepts and Contexts

Fifth Edition

James Stewart

McMaster University
and
University of Toronto

Stephen Kokoska

Bloomsburg University

Cengage

Australia • Brazil • Canada • Mexico • Singapore • United Kingdom • United States

Cengage

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James Stewart, Stephen Kokoska

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IP Project Manager: Carly Belcher

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Cengage
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Contents

Preface

When the first edition of this book appeared over 20 years ago, there was a lot of discussion centered on calculus reform. Many mathematics departments were divided on issues including the use of technology, conceptual understanding versus procedural practice, and the role of discovery learning. Since then, the Advanced Placement® Calculus program has embraced calculus reform, and reformers and traditionalists have realized that they have a common goal: to enable students to understand and appreciate calculus.

The first four editions were intended to be a synthesis of reform and traditional approaches to calculus instruction. In the fifth edition, we continue this approach by emphasizing conceptual understanding through graphical, verbal, numerical, and algebraic approaches. We would like students to learn important problem-solving skills, and to see both the practical power of calculus and the intrinsic beauty of the subject.

The principal way in which this book differs from the more traditional calculus textbooks is that it is more streamlined. For example, there is no complete chapter on techniques of integration; we do not prove as many theorems; and the material on transcendental functions and on parametric equations is interwoven throughout the book instead of being treated in separate chapters. Instructors who prefer a more complete coverage of traditional calculus topics should consider *Calculus*, Ninth Edition, and Calculus: *Early Transcendentals*, Ninth Edition.

What's New in the Fifth Edition?

The changes in the fifth edition include a more conversational tone with an uncluttered presentation, all focused on conceptual understanding through the development of problem-solving skills. Here are some of the specific improvements that we have incorporated into this edition:

- *A Closer Look* feature provides straightforward itemized explanations of important concepts. Students will find these easy to read and to connect with the relevant theory.
- Marginal notes titled *Common Error* remind students of common errors and reinforce the proper solution technique.
- More detailed, guided solutions to examples include explanations for most steps (easy to read, in a different color, right justified with the appropriate step). This makes it easier for the student to follow the logical steps to a solution and to apply problem-solving skills to exercises.
- Wherever possible, sections are divided into appropriate subsections, smaller pieces, to accommodate the way students read and learn today.
- All graphs have been redrawn to include more detail and every figure has an appropriate caption to easily link with the appropriate idea.

- Each chapter begins with a real-world situation that introduces the material.
- The data in examples and exercises have been updated to be more timely.
- Section 1.4, “Graphing Calculators and Computers,” has been eliminated.
- Former Section 2.8, “What Does f' Say About f,” has been incorporated into Section 4.3, “Derivatives and the Shapes of Curves.”
- New WebAssign problem types and learning resources build student problem-solving skills and conceptual understanding. These include automatically graded proof problems, Expanded Problems, Explore It interactive learning modules, and an eTextbook with Media Index and Student Solutions Manual.

Features

Conceptual Exercises

The exercises include various types of problems to foster conceptual understanding. Some exercises sets begin with questions that ask for an explanation of some of the basic concepts presented in the section. See, for example, the first few exercises in Sections 2.2, 2.4, 2.5, 5.3, 8.2, 11.2, and 11.3. These problems might be used as a basis for classroom discussions. Similarly, review sections begin with a Concepts and Vocabulary section and a True-False Quiz. Other Exercises test conceptual understanding through graphs and tables. See, for example, Exercises 1.7.22–25, 2.6.19, 2.7.39–42; 45–48, 3.8.5–6, 5.2.65–67, 7.1.12–14, 8.7.2, 10.2.1, 10.3.37–41, 11.1.1–2, 11.1.12–22, 11.3.3–10, 11.6.1–3, 11.7.3–4, 12.1.7–12, 13.1.13–22, 13.2.18–19, and 13.3.1, 2, 13.

Another type of exercise uses verbal description to test conceptual understanding. See, for example, Exercises 2.4.11, 2.7.75, 4.3.80, 4.3.84–85, and 5.10.69. Other exercises combine and compare graphical, numerical, and algebraic approaches; see Exercises 2.5.54–55, 2.5.63, 3.8.27, and 7.5.4.

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises and skill-development problems to more challenging problems involving applications and proofs.

Real-World Data

Everyone involved with this writing project has spent a great deal of time looking in libraries, contacting companies and government agencies, and searching the Internet for interesting real-world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises are associated with functions defined by numerical data given in a table or graphically. See, for example, Figure 1.1 in Section 1.1 (the rate of water usage in New York City during the 2018 Super Bowl), Exercise 5.1.16 (the velocity of a car racing at the Daytona International Speedway), Exercise 5.1.18 (the velocity of a pod in the SpaceX Hyperloop), Figure 5.40 (San Francisco power consumption), Example 5.9.5 (data traffic on Internet links), and Example 9.6.3 (wave heights).

Functions of two variables are illustrated by a table of values of the wind-chill index as a function of the wind speed and the air temperature (Example 11.1.1). Partial derivatives are introduced in Section 11.3 by examining a column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. This example is considered again in connection with linear approximations (Example 11.4.3). Directional derivatives are introduced in Section 11.6 by using a temperature contour map to estimate the rate of change of temperature at Boston in a northwest direction. Double integrals are used to estimate the average snowfall in Colorado during the 2020–2021 winter (Example 12.1.4). Vector fields are introduced in Section 13.1 by depictions of actual velocity vector fields showing wind patterns and ocean currents.

Projects

One way to involve students and to help make them active learners is to have them work (perhaps in groups) on extended projects that lead to a feeling of substantial accomplishment when completed. Applied Projects involve applications that are designed to appeal to the imagination of students. The project after Section 3.1 asks students to design the first ascent and drop for a roller coaster. The project after Section 11.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity. Laboratory Projects involve technology; the project following Section 3.4 shows how to use Bezier curves to design shapes that represent letters for a laser printer. Writing Projects ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance. Suggested references are supplied. Discovery Projects anticipate results to be discussed later or cover optional topics (hyperbolic functions) or encourage discovery through pattern recognition (see the project following Section 5.8). Others explore aspects of geometry: tetrahedra (after Section 9.4), hyperspheres (after Section 12.7), and intersections of three cylinders (after Section 12.8). Additional projects can be found in the Instructor's Guide (see, for instance, Group Exercise 5.1: Position from Samples) and also in the CalcLabs supplements.

Rigor

There are fewer proofs included in this text as compared with more traditional calculus books. However, it is still worthwhile to expose students to the idea of proof and to make a clear distinction between a proof and a plausibility argument using, for example, technology (a graph or a table of values). The important thing is to show how to reach a conclusion that seems less obvious from something that seems more obvious. A good example is the use of the Mean Value Theorem to prove the Evaluation Theorem (Part 2 of the Fundamental Theorem of Calculus). Note that we have chosen not to prove the convergence tests but rather present intuitive arguments that they are true.

Problem Solving

Problem solving is perhaps the most difficult concept to teach and learn. Students frequently have difficulty solving problems in which there is no single well-defined procedure or technique for obtaining the final answer. It seems that no one has improved very much on George Polya's four-stage problem-solving strategy and, accordingly, a version of his problem-solving principles is included at the end of Chapter 1. These principles are applied, both explicitly and implicitly, throughout the book. At the end of other chapters, there are sections called Focus on Problem Solving, which feature examples of how to approach challenging calculus problems. The varied problems in these sections are selected using the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." These challenging problems might be used on assignments and tests, but consider grading them in a different way. One might reward a student significantly for presenting ideas toward a solution and for recognizing which problem-solving principles are relevant.

Technology

Graphing calculators and computers are powerful tools that allow us to explore problems, discover concepts, and confirm solutions. However, it is even more important to understand clearly the concepts that underlie the results and images on the screen. We assume that the student has access to either a graphing calculator or a computer algebra system. But technology doesn't make pencil and paper obsolete. Hand calculations and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where the use of technology is appropriate.

Cengage WebAssign

WebAssign: webassign.com

This Fifth Edition is available with WebAssign, a fully customizable online solution for STEM disciplines from Cengage. WebAssign includes homework, an interactive mobile eBook, videos, tutorials, and Explore It interactive learning modules. Instructors can decide what type of help students can access, and when, while working on assignments. The patented grading engine provides unparalleled answer evaluation, giving students instant feedback, and insightful analytics highlight exactly where students are struggling. For more information, visit webassign.com.

Stewart Website

Visit StewartCalculus.com for these additional materials:

- Homework Hints
- Algebra and Analytic Geometry Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to recommended historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Rotation of Axes, Formulas for the Remainder Theorem in Taylor Series, Second-Order Differential Equations
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources

Content

Diagnostic Tests

The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

A Preview of Calculus

This is an overview of the subject and includes a list of questions to motivate the study of calculus.

1 • Functions and Models

Multiple representations of functions are emphasized throughout the text: verbal, numerical, graphical, and algebraic. A discussion of mathematical models leads to a review of the standard functions, including exponential and logarithmic functions, from these four points of view. Parametric curves are introduced in the first chapter, partly, so that curves can be drawn easily, with technology, whenever needed throughout the text. This early placement also enables tangents to parametric curves to be treated in Section 3.4 and graphing such curves to be covered in Section 4.4.

2 • Limits

The material on limits is motivated by a discussion of the tangent line and instantaneous velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Note that the precise definition of a limit is provided in Appendix D for those who wish to cover this concept. It is important to carefully consider Sections 2.6 and 2.7, which deal with derivatives and rates of change, before the differentiation rules are covered in Chapter 3. The examples and exercises in these sections explore the meanings of derivatives in various contexts.

3 • Differentiation Rules

All of the rules for differentiating basic functions are presented in this chapter. There are many applied examples and exercises in which students are asked to explain the meaning of the derivative in the context of the problem. Optional topics (hyperbolic functions, an early introduction to Taylor polynomials) are explored in Discovery and Laboratory Projects. A full treatment of hyperbolic functions is available to instructors on the website.

4 • Applications of Differentiation

This chapter begins with a section on related rates. Then, the basic facts concerning extreme values and shapes of curves are derived using the Mean Value Theorem as the starting point. The interaction between technology and calculus is discussed and illustrated, and there are a wide variety of optimization problems presented. Indeterminate forms are addressed, Newton's method is presented, and a discussion of antiderivatives prepares students for Chapter 5.

5 • Integrals

The area problem and the distance problem serve to motivate the definite integral. Subintervals of equal width are used in order to make the definition of a definite integral easier to understand. Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables. There is no separate chapter on techniques of integration, but substitution and integration by parts are covered here and other methods are treated briefly. Partial fractions are given full treatment in Appendix G. The use of computer algebra systems is discussed in Section 5.8.

6 • Applications of Integration

General methods, not formulas, are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral. There are lots of applications in this chapter, probably too many to cover in any one course. We hope you will select applications that you and your students enjoy. Some instructors like to cover polar coordinates, in Appendix H, here. Others prefer to defer this topic until it is needed in a third semester calculus course, with Section 9.7 or just before Section 12.4.

7 • Differential Equations

Modeling is the theme that unifies this introductory treatment of differential equations. Slope fields and Euler's method are presented before separable equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. Predator-prey models are used to illustrate systems of differential equations.

8 • Infinite Sequences and Series

Tests for the convergence of series are considered briefly, with intuitive rather than formal justifications. Numerical estimates of sums of series are based on the test used to prove convergence. The emphasis is on Taylor series and polynomials, their applications to physics, and error estimates.

9 • Vectors and the Geometry of Space

The dot product and cross product of vectors are given geometric definitions, motivated by work and torque, before the algebraic expressions are derived. To facilitate the discussion of surfaces, functions of two variables and their graphs are introduced here.

10 • Vector Functions

The calculus of vector functions is used to prove Kepler's First Law of planetary motion, with the proofs of the other laws left as a project. Since parametric curves were introduced in Chapter 1, parametric surfaces are introduced as soon as possible, namely, in this chapter. We think an early familiarity with such surfaces is desirable, especially with the capability of computers to produce their graphs. Then tangent planes and areas of parametric surfaces can be discussed in Sections 11.4 and 12.6.

11 • Partial Derivatives

Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, partial derivatives are introduced by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. Directional derivatives are estimated from contour maps of temperature, pressure, and elevation.

12 • Multiple Integrals

Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute probabilities, areas of parametric surfaces, volumes of hyperspheres, and the volume of intersection of three cylinders.

13 • Vector Fields

Vector fields are introduced through pictures of velocity fields showing wind patterns and ocean currents. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

Acknowledgments

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James Stewart
Stephen Kokoska

Instructor Resources

Cengage WebAssign

Built by educators, WebAssign from Cengage is a fully customizable online solution for STEM disciplines. WebAssign includes the flexibility, tools, and content you need to create engaging learning experiences for your students. The patented grading engine provides unparalleled answer evaluation, giving students instant feedback, and insightful analytics highlight exactly where students are struggling. For more information, visit *webassign.com.*

Additional instructor resources for this product are available online at the Cengage Instructor Center—an all-in-one resource for class preparation, presentation, and testing. Instructor resources available for download from the Cengage Instructor Center include:

Instructor's Manual

Includes activities and assessments correlated by learning objectives, chapter and section outline, key formulas and terms with definitions, ideas for student collaboration and class discussions, and more.

Solutions and Answer Guide

Provides answers and solutions to all exercises in the text, including exercises in the end-of-chapter Review and Focus on Problem Solving.

Cengage Testing Powered by Cognero®

A flexible online system that allows you to author, edit, and manage test bank content online. You can create multiple tests in an instant and deliver them from your LMS, or export to printable PDF or Word format for in-class assessment.

PowerPoint® Slides

The PowerPoint® slides are ready to use, visual outlines of each section that can be easily customized for your lectures. Presentations include activities, examples, and opportunities for student engagement and interaction.

Transition Guide

Highlights the content changes from the previous edition to the new edition, including exercise correlations.

Guide to Online Teaching

Provides technological and pedagogical considerations and tips for teaching a Calculus course online.

Educator's Guide

Offers suggested content and activities for Cengage WebAssign—like videos and assignments—that you can integrate into your course to help boost engagement and outcomes.

Sign up or sign in at **www.cengage.com** to search for and access this product and its online resources.

Student Resources

Cengage WebAssign

Prepare for class with confidence using WebAssign from Cengage. This online learning platform, which includes an interactive eBook, encourages practice, so that you truly absorb what you learn and prepare better for tests. Videos and tutorials walk you through concepts and deliver instant feedback and grading, so you always know where you stand in class. Focus your study time and get extra practice where you need it most. Study smarter with WebAssign! Ask your instructor today how you can get access to WebAssign, or learn about self-study options at *webassign.com.*

Student Solutions

Provides step-by-step solutions for all odd-numbered exercises in the text. The Student Solutions are provided in the eBook at no additional cost.

Additional student resources are available online. Sign up or sign in at **www.cengage.com** to search for and access this product and its online resources.

To the Student

Reading a calculus book is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once to understand it. It's a good idea to keep a pencil, paper, and graphing calculator handy to check, or try, a calculation or sketch a diagram or graph.

Some students don't read the text at first; they start by trying the homework problems and read the text only when they get stuck on an exercise. We think a better plan is to read and understand a section of the text before attempting the exercises. In particular, you should carefully study the definitions and understand the exact meanings of terms. And before you read an example, we suggest that you cover up the solution and try solving the problem yourself. You will learn a lot more from looking at the solution this way.

This text is designed to teach conceptual understanding through problem-solving skills and pattern recognition, and to train you to think logically. Learn to write solutions to the exercises in a connected, step-by-step fashion, using good communication and proper notation.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix J. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several forms in which to express a numerical or algebraic answer, so if your answer differs from the one in the back of the book, don't immediately assume you are wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and your answer is $1/(1 + \sqrt{2})$, then you are right, and rationalizing the denominator will show that the answers are equivalent.

The Stewart website is a companion to this text and provides various resources to help you succeed. For example, Homework Hints for representative exercises ask questions that allow you to make progress toward a solution without actually giving you the answer. You will need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can reveal the next hint. There is also an algebra review, some drill exercises to reinforce techniques, and several challenge problems.

We hope that you will keep this book for reference after you finish the course. Because you may forget some of the specific details of calculus, this book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer. Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. We hope that you will discover that calculus is not only useful but also intrinsically beautiful.

James Stewart
Stephen Kokoska

About the Author

Steve received his undergraduate degree from Boston College and his M.S. and Ph.D. from the University of New Hampshire. His initial research interests included the statistical analysis of cancer chemoprevention experiments. He has published a number of research papers in mathematics journals, including *Biometrics, Anticancer Research,* and *Computer Methods and Programs in Biomedicine*; presented results at national conferences; and written several books. He has been awarded grants from the National Science Foundation, the Center for Rural Pennsylvania, and the Ben Franklin Program.

Steve is a longtime consultant for the College Board and has conducted workshops in Brazil, the Dominican Republic, Singapore, and China. He was the AP Calculus Chief Reader for four years, has been involved with calculus reform and the use of technology in the classroom, and recently published an AP Calculus text with James Stewart. He taught at Bloomsburg University for 30 years and has served as Director of the Honors Program.

Steve believes in teaching conceptual understanding through the development of problem-solving skills and pattern recognition. He is still involved with the AP Calculus program and co-hosts the webinar Monday Night Calculus sponsored by Texas Instruments.

Steve's uncle, Fr. Stanley Bezuszka, a Jesuit and professor at Boston College, was one of the original architects of the so-called new math in the 1950s and 1960s. He had a huge influence on Steve's career. Steve helped Fr. B. with text accuracy checks, as a teaching assistant, and even writing projects through high school and college. Steve learned about the precision, order, and elegance of mathematics and developed an unbounded enthusiasm to teach.

Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following diagnostic tests will help you assess your proficiency in these subjects. After taking each test, you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$ (b) $(3a^3b^3)(4ab^2)^2$ (c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5} - 2}$ (b) $\dfrac{\sqrt{4 + h} - 2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x + 1} = \dfrac{2x - 1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leq 17$
(b) $x^2 < 2x + 8$
(c) $x(x-1)(x+2) > 0$
(d) $|x - 4| < 3$
(e) $\dfrac{2x-3}{x+1} \leq 1$

10. State whether each equation is true or false.

(a) $(p+q)^2 = p^2 + q^2$
(b) $\sqrt{ab} = \sqrt{a}\sqrt{b}$
(c) $\sqrt{a^2 + b^2} = a + b$
(d) $\dfrac{1 + TC}{C} = 1 + T$
(e) $\dfrac{1}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$
(f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a-b}$

Answers to Diagnostic Test A: Algebra

1. (a) 81 (b) -81 (c) $\dfrac{1}{81}$
(d) 25 (e) $\dfrac{9}{4}$ (f) $\dfrac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x-5)(2x+5)$ (b) $(2x-3)(x+4)$
(c) $(x-3)(x-2)(x+2)$ (d) $x(x+3)(x^2-3x+9)$
(e) $3x^{-1/2}(x-1)(x-2)$ (f) $xy(x-2)(x+2)$

5. (a) $\dfrac{x+2}{x-2}$ (b) $\dfrac{x-1}{x-3}$
(c) $\dfrac{1}{x-2}$ (d) $-(x+y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4+h}+2}$

7. (a) $\left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ (b) $2(x-3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\dfrac{2}{3}, \dfrac{22}{3}$
(g) $\dfrac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4)$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you have had difficulty with these problems, you may wish to consult the Review of Algebra on the website www.stewartcalculus.com.

B Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point $(2, -5)$ and

(a) has slope -3

(b) is parallel to the x-axis

(c) is parallel to the y-axis

(d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
(a) Find the slope of the line that contains A and B.
(b) Find an equation of the line that passes through A and B. What are the intercepts?
(c) Find the midpoint of the segment AB.
(d) Find the length of the segment AB.
(e) Find an equation of the perpendicular bisector of AB.
(f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.

(a) $-1 \le y \le 3$ (b) $|x| < 4$ and $|y| < 2$ (c) $y < 1 - \frac{1}{2}x$

(d) $y \ge x^2 - 1$ (e) $x^2 + y^2 < 4$ (f) $9x^2 + 16y^2 = 144$

Answers to Diagnostic Test B: Analytical Geometry

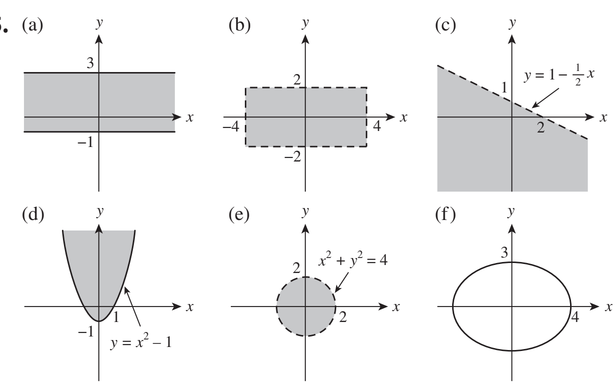
1. (a) $y = -3x + 1$
(b) $y = -5$
(c) $x = 2$
(d) $y = \frac{1}{2}x - 6$

2. $(x + 1)^2 + (y + 4)^2 = 52$

3. Center $(3, -5)$, radius 5

4. (a) $-\frac{4}{3}$
(b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
(c) $(-1, -4)$
(d) 20
(e) $3x - 4y = 13$
(f) $(x + 1)^2 + (y + 4)^2 = 100$

5.



If you have had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendix B.

C Diagnostic Test: Functions

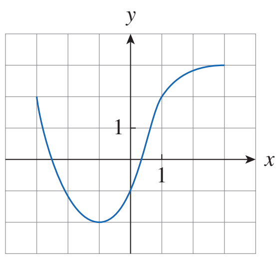


Figure for Problem 1

1. The graph of a function f is given at the left.

(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.

(a) $f(x) = \dfrac{2x+1}{x^2+x-2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$ (c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$

4. How are graphs of the functions obtained from the graph of f?

(a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x-3) + 2$

5. Without using a calculator, make a rough sketch of the graph.

(a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = (x-2)^3 + 3$
(d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
(g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.

(a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

Answers to Diagnostic Test C: Functions

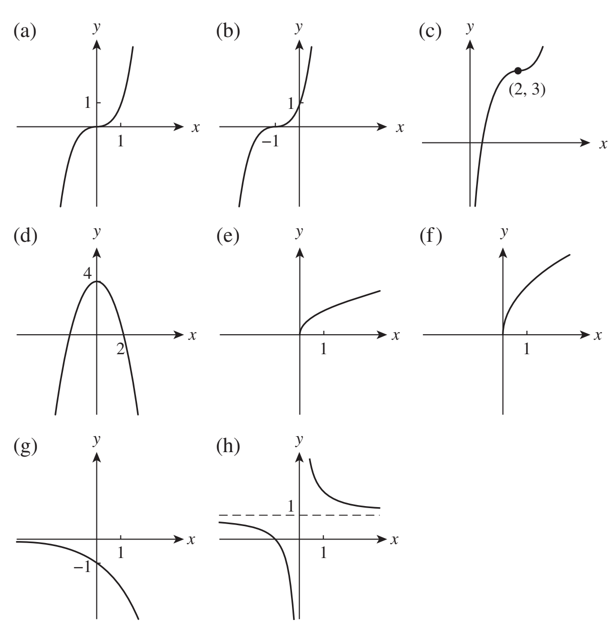
1. (a) -2 (b) 2.8
(c) $-3,\ 1$ (d) $-2.5,\ 0.3$
(e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
(b) $(-\infty, \infty)$
(c) $(-\infty, -1] \cup [1, 4]$

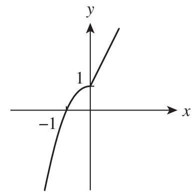
4. (a) Reflect about the x-axis.
(b) Stretch vertically by a factor of 2, then shift 1 unit downward.
(c) Shift 3 units to the right and 2 units upward.

5.



6. (a) $-3, 3$

(b)



7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = 8x - 21$

If you have had difficulty with these problems, you should look at Sections 1.1–1.3 of this book.

D Diagnostic Test: Trigonometry

1. Convert from degrees to radians.

(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.

(a) $\dfrac{5\pi}{6}$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of $30°$.

4. Find the exact values.

(a) $\tan\left(\dfrac{\pi}{3}\right)$ (b) $\sin\left(\dfrac{7\pi}{6}\right)$ (c) $\sec\left(\dfrac{5\pi}{3}\right)$

5. Express the lengths a and b in the figure in terms of θ.

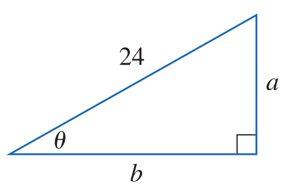


Figure for Problem 5

6. If $\sin x = \dfrac{1}{3}$ and $\sec y = \dfrac{5}{4}$, where x and y lie between 0 and $\dfrac{\pi}{2}$, evaluate $\sin(x + y)$.

7. Prove the identities.

(a) $\tan\theta \ \sin\theta + \cos\theta = \sec\theta$

(b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \le x \le 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

Answers to Diagnostic Test D: Trigonometry

1. (a) $\dfrac{5\pi}{3}$ (b) $\dfrac{-\pi}{10}$

2. (a) $150°$ (b) $\dfrac{360°}{\pi} \approx 114.6°$

3. 2π cm

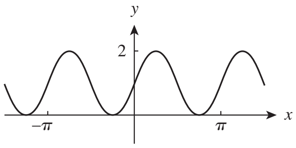
4. (a) $\sqrt{3}$ (b) $-\dfrac{1}{2}$ (c) 2

5. (a) $24\sin\theta$ (b) $24\cos\theta$

6. $\dfrac{1}{15}(4 + 6\sqrt{2})$

8. $0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}, 2\pi$

9.



If you have had difficulty with these problems, you should look at Appendix C of this book.

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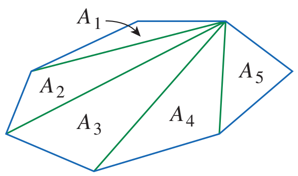
A Preview of Calculus

Mark Twain wrote, "If you don't like the weather in New England now, just wait a few minutes." Indeed, the weather in New England changes quickly, and calculus is about the study of change, instantaneous change. Complex mathematical models can be used to predict changes in weather, and calculus plays an important role in these analyses.

Calculus is fundamentally different from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with the long-run, or limiting, behavior of certain expressions. For that reason it may be useful to have an overview of the subject before beginning its intensive study. This preview provides a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.



$A = A_1 + A_2 + A_3 + A_4 + A_5$

Figure 1
The area of the polygon is the sum of the areas of the triangles.

The Area Problem

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

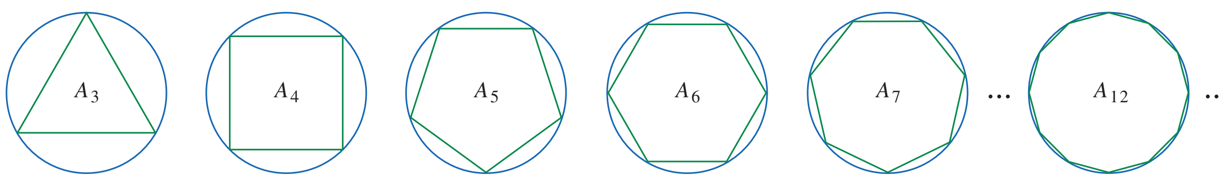


Figure 2
Regular inscribed polygons used to find the area of a circle.

Let A_n be the area of the inscribed polygon with n sides. As n increases, it appears that A_n becomes closer and closer to the area of the circle. We say that the area of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} A_n$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (5th century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 5 to find areas of regions of the type shown in Figure 3. We will approximate the desired area A by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate A as the limit of these sums of areas of rectangles.

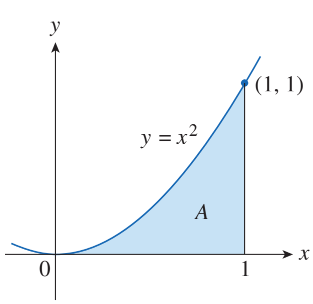


Figure 3
Let A be the area of the shaded region.

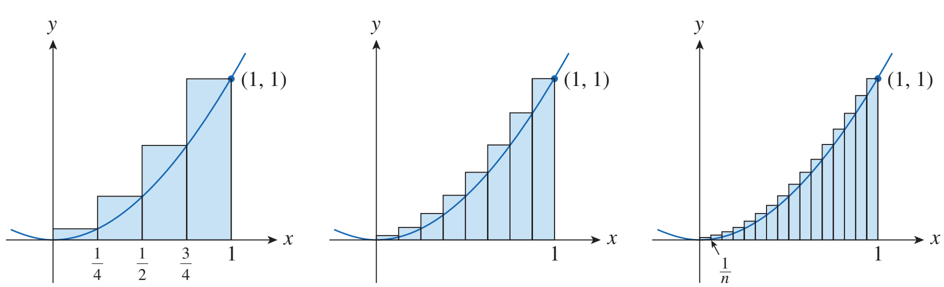


Figure 4
Approximate the area A by areas of rectangles.

The area problem is the central problem in the branch of calculus called *integral calculus.* The techniques that we will develop in Chapter 5 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

The Tangent Line Problem

Consider the problem of trying to find an equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P. (We will give a precise definition of a tangent line in Chapter 2. For now you can think of it as a line that just touches the curve at P as in Figure 5.) Since we know that the point P lies on the tangent line, we can find the equation of t if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on t. To solve this problem, we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ. From Figure 6 we see that

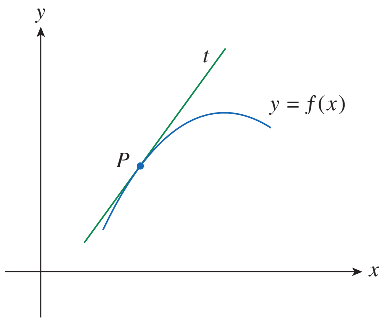


Figure 5
The tangent line to the graph of $y = f(x)$ at the point P.

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} \tag{1}$$

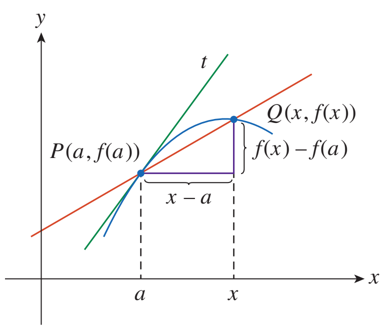


Figure 6
The secant line PQ.

Now imagine that Q moves along the curve toward P as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve. Since x approaches a as Q approaches P, we could also use Equation 1 to write

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

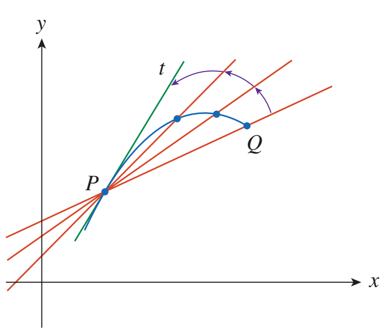


Figure 7
Secant lines approaching the tangent line.

Specific examples of this procedure will be given in Chapter 2.

The tangent line problem has given rise to the branch of calculus called *differential calculus*, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent line problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent line problem and the area problem are inverse problems in a sense that will be described in Chapter 5.

Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information really mean? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following table:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	9	24	42	71

As a first step toward finding the velocity after 2 seconds have elapsed, we find the average velocity during the time interval $2 \le t \le 4$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{42 - 9}{4 - 2} = 16.5 \text{ ft/s}$$

Similarly, the average velocity in the time interval $2 \le t \le 3$ is

$$\text{average velocity} = \frac{24 - 9}{3 - 2} = 15 \text{ ft/s}$$

It seems reasonable that the velocity at the instant $t = 2$ can't be much different from the average velocity during a short time interval starting at $t = 2$. So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following table:

t	2.0	2.1	2.2	2.3	2.4	2.5
d	9.00	10.02	11.16	12.45	13.96	15.80

Then we can compute, for instance, the average velocity over the time interval [2, 2.5]:

$$\text{average velocity} = \frac{15.80 - 9.00}{2.5 - 2} = 13.6 \text{ ft/s}$$

The results of such calculations are shown in the following table:

Time interval	[2, 3]	[2, 2.5]	[2, 2.4]	[2, 2.3]	[2, 2.2]	[2, 2.1]
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly $t = 2$ is about 10 ft/s. In Chapter 2 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

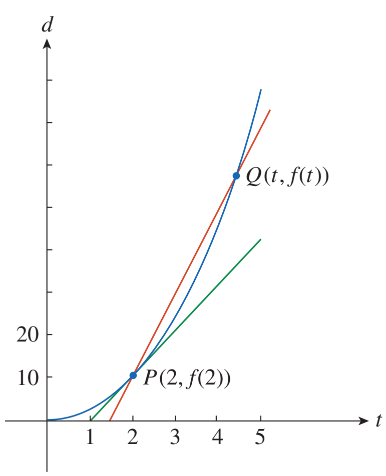


Figure 8
A graphical illustration of the motion of the car.

Figure 8 shows a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write $d = f(t)$, then $f(t)$ is the number of feet traveled after t seconds. The average velocity in the time interval $[2, t]$ is

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

which is the same expression as the slope of the secant line PQ in Figure 8. The velocity v when $t = 2$ is the limiting value of this average velocity as t approaches 2; that is,

$$v = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at P.

Thus, when we solve the tangent line problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

The Limit of a Sequence

In the 5th century BC, the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: suppose that Achilles starts at position a_1 and the tortoise starts at position t_1. (See Figure 9.) When Achilles reaches the point $a_2 = t_1$, the tortoise is farther ahead at position t_2. When Achilles reaches $a_3 = t_2$, the tortoise is at t_3. This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.

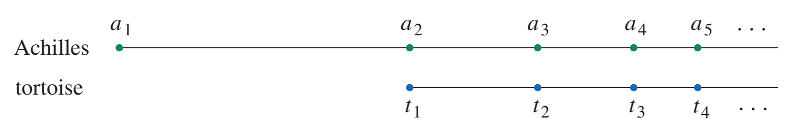


Figure 9
A graphical illustration of the race between Achilles and a tortoise.

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles $(a_1, a_2, a_3, \ldots)$ or the successive positions of the tortoise $(t_1, t_2, t_3, \ldots)$ form what is known as a sequence.

In general, a sequence, denoted $\{a_n\}$, is a set of numbers written in a definite order. For instance, the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

can be described by using the following formula for the nth term:

$$a_n = \frac{1}{n}$$

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence $a_n = 1/n$ are becoming closer and closer to 0 as n increases. In fact, we can find terms as small as we please by making n large enough. We say that the limit of the sequence is 0, and we indicate this behavior by writing

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

is used if the terms a_n approach the number L as n becomes large. This means that the numbers a_n can be made as close as we like to the number L by taking n sufficiently large.

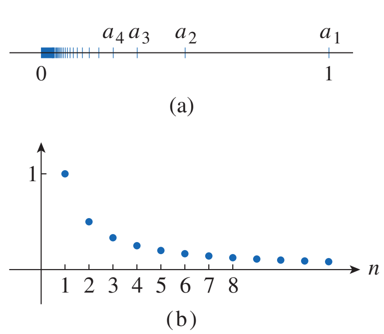


Figure 10
Two ways to visualize a sequence.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$a_1 = 3.1$$
$$a_2 = 3.14$$
$$a_3 = 3.141$$
$$a_4 = 3.1415$$
$$a_5 = 3.14159$$
$$a_6 = 3.141592$$
$$a_7 = 3.1415926$$
$$\vdots$$

then

$$\lim_{n\to\infty} a_n = \pi$$

The terms in this sequence are rational approximations to π.

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences $\{a_n\}$ and $\{t_n\}$, where $a_n < t_n$ for all n. It can be shown that both sequences have the same limit:

$$\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

The Sum of a Series

Another of Zeno's paradoxes involves a person in a room attempting to walk to a wall. As Aristotle indicated, in order to do so, they would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended. (See Figure 11.)

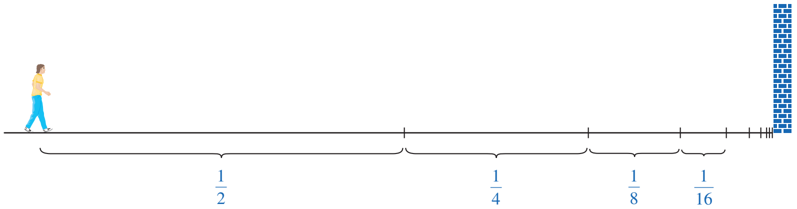


Figure 11
A visualization of a walk to the wall.

Of course, we know that the person can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots \tag{3}$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol $0.\overline{3} = 0.3333\ldots$ means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots = \frac{1}{3}$$

More generally, if d_n denotes the nth digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} + \cdots$$

Therefore, some infinite sums, or infinite series as they are called, have a meaning. But we must carefully define the sum of a series.

Returning to the series in Equation 3, the sum of the first n terms of the series is called the partial sum and is denoted by s_n. Thus

$$\begin{aligned}
s_1 &= \frac{1}{2} = 0.5\\
s_2 &= \frac{1}{2} + \frac{1}{4} = 0.75\\
s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875\\
s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.9375\\
s_5 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96875\\
s_6 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.984375\\
s_7 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 0.9921875\\
&\vdots\\
s_{10} &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024} \approx 0.99902344\\
&\vdots\\
s_{16} &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{16}} \approx 0.99998474
\end{aligned}$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking n large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum s_n as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

In other words, the reason the sum of the series is 1 is that

$$\lim_{n\to\infty} s_n = 1$$

In Chapter 8 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent line to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

After Sir Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast oil prices rise or fall, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a wide variety of other areas. We will explore some of these uses of calculus in this book.

In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

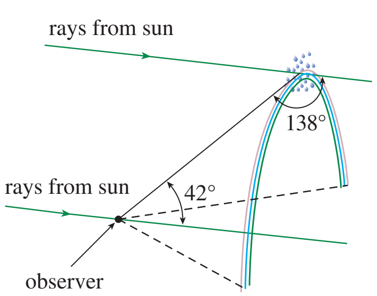
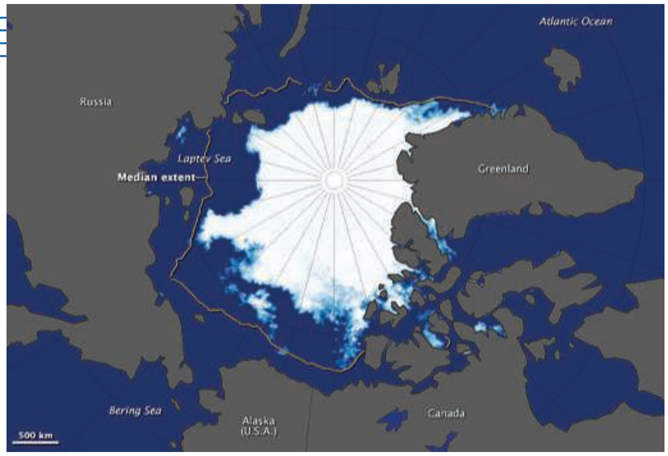


Figure 12
The elevation up to the highest point in a rainbow.

1. How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is 42°? (See page 313.)
2. How can we explain the shapes of cans on supermarket shelves? (See page 372.)
3. Where is the best place to sit in a movie theater? (See page 574.)
4. How far away from an airport should a pilot start descent? (See page 237.)
5. How can we fit curves together to design shapes to represent letters on a laser printer? (See page 236.)
6. Where should an infielder position themself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 653.)
7. Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 643.)
8. How can we explain the fact that planets and satellites move in elliptical orbits? (See page 886.)
9. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production? (See page 994.)
10. If a marble, a squash ball, a steel bar, and a lead pipe roll down a slope, which of them reaches the bottom first? (See page 1070.)



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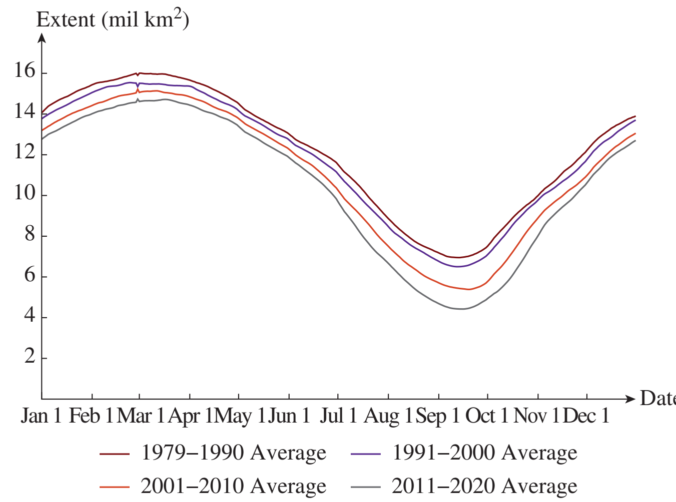

Sea ice is simply frozen water, but the extent, or total area, of sea ice is related to the climate in the polar regions and the entire world. Sea ice reflects sunlight. Therefore, a large sea ice extent tends to keep a region cool. A small sea ice extent exposes dark water, which absorbs sunlight. The ocean tends to heat up, which warms the surrounding area. Scientists have studied the change in sea ice extent over the past four decades. Often a graph is the best way to represent this information because it conveys many significant features at a glance. For example, a graph of the average sea ice extent over four different decades is shown in the accompanying figure (National Snow and Ice Data Center). Using this information and some calculus, we will soon be able to estimate the rate of change of sea ice extent at any time of even the time of year at which the sea ice is a maximum or a minimum.

1 Functions and Models

Contents

The fundamental objects that we develop and utilize in calculus are functions. This chapter prepares the way for calculus by presenting and reviewing the basic concepts concerning functions, their graphs, and methods of transforming and combining them. Throughout this text, we stress that a function can be represented in a variety of ways: by an equation, in a table, by a graph, or in words (the Rule of Four). We look at main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena. We will also see that parametric equations provide the best method for graphing certain types of curves.